Internal precategories relative to split epimorphisms

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Abstract. For a given category B we are interested in studying internal categorical structures in B. This work is the starting point, where we consider reflexive graphs and precategories (i.e., for the purpose of this note, a simplicial object truncated at level 2). We introduce the notions of reflexive graph and precategory relative to split epimorphisms. We study the additive case, where the split epimorphisms are "coproduct projections", and the semi-additive case where split epimorphisms are "semi-direct product projections". The result is a generalization of the well known equivalence between precategories and 2-chain complexes. We also consider an abstract setting, containing, for example, strongly unital categories.

1. Introduction

A internal reflexive graph in the category Ab, of abelian groups, is completely determined, up to an isomorphism, by a morphism $h : X \longrightarrow B$ and it is of the following form

$$\begin{tikzcd} X \oplus B \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "{[h\,1]}"'] & B \arrow[l, "\iota_2" description] \end{tikzcd} .$$

An internal precategory (i.e., for the purpose of this work, a simplicial object truncated at level 2) is, in the first place, determined by a diagram

$$\begin{tikzcd} Y \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "u"'] & X \arrow[l, "b" description] \arrow[r, "h"] & B \end{tikzcd}$$

such that

$$ab = 1 = ub\ ,\ ha = hu,$$

and later, with a further analysis, it simplifies to a 2-chain (see [**5**] and [**9**] for more general results on this topic), i.e.,

$$Z \xrightarrow{t} X \xrightarrow{h} B \quad , \quad ht = 0.$$

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And it is always of the following form

$$\begin{tikzcd}[column sep=large] (Z \oplus X) \oplus (X \oplus B) \arrow[r, shift left=6, "\pi_2"] \arrow[r, shift left=3, leftarrow, "\iota_2"] \arrow[r, "{[\iota_1[t\,1],1]}"] \arrow[r, shift right=3, leftarrow, "\iota_2 \oplus \iota_2"'] \arrow[r, shift right=6, "\pi_2 \oplus [h\,1]"'] & X \oplus B \arrow[r, shift left=3, "\pi_2"] \arrow[r, leftarrow, "\iota_2"] \arrow[r, shift right=3, "{[h\,1]}"'] & B \end{tikzcd} .$$

The same result holds for arbitrary additive categories with kernels. In this work we will be interested in answering the following question: "what is the more general setting where one can still have similar results?".

An old observation of G. Janelidze, says that "since every higher dimensional categorical structure is obtained from an n-simplicial object; and since a simplicial object is build up from split epis; and, since in Ab, every split epi is simply a biproduct projection, then it is expected that, when internal to Ab, all the higher dimensional structures reduce to categories of presheaves". We use this observation as a motivation for the study of internal categorical structures restricted to a given subclass of split epis.

In particular in this work we will be interested in the study of the notion of internal reflexive graph (1-simplicial object) and internal precategory (2-simplicial object) relative to a given subclass of split epis, such as for example coproduct projections (in a pointed category with coproducts), or product projections, or semidirect product projections, or etc.

In some cases the given subclass is saturated (in the words of D. Bourn), as it happens for example in an additive category with kernels for the subclass of biproduct projections. However, in general, this is not the case; nevertheless, in some cases, interesting notions do occur.

Take for example the category of pointed sets and the class of coproduct projections, that is, consider only split epis of the form

$$\begin{tikzcd} X \sqcup B \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=1, leftarrow, "\iota_2"'] & B \end{tikzcd} ;$$

It follows that a reflexive graph relative to coproduct projections is completely determined by a morphism

$$h : X \longrightarrow B$$

and it is of the form

$$\begin{tikzcd} X \sqcup B \arrow[r, shift left=3, "{[0\,1]}"] \arrow[r, leftarrow, "\iota_2"] \arrow[r, shift right=3, "{[h\,1]}"'] & B \end{tikzcd} ;$$

While a precategory is determined by a diagram

$$\begin{tikzcd} Y \arrow[r, shift left=3, "a"] \arrow[r, leftarrow, "b"] \arrow[r, shift right=3, "u"'] & X \arrow[r, "h"] & B \end{tikzcd} \quad , \quad ab = 1 = ub \ , \ ha = hu,$$

and it is of the form

$$\begin{tikzcd}[column sep=large] Y \sqcup (X \sqcup B) \arrow[r, shift left=6, "{[0\,1]}"] \arrow[r, shift left=3, leftarrow, "\iota_2"] \arrow[r, "{[\iota_1 u, 1]}"] \arrow[r, shift right=3, leftarrow, "b \sqcup \iota_2"'] \arrow[r, shift right=6, "a \sqcup [h\,1]"'] & X \sqcup B \arrow[r, shift left=3, "{[0\,1]}"] \arrow[r, leftarrow, "\iota_2"] \arrow[r, shift right=3, "{[h\,1]}"'] & B \end{tikzcd}$$

where the key factor for this result to hold is the fact that ι_1 is the kernel of $[0\ 1]$.

Furthermore, every such reflexive graph $(h : X \longrightarrow B)$ may be considered as a precategory,

$$\begin{tikzcd} X \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & X \arrow[l, "1" description] \arrow[r, "h"] & B \end{tikzcd} ,$$

and it is a internal category if the kernel of h is trivial, which is the same as saying that the following square

$$\begin{tikzcd} X \sqcup (X \sqcup B) \arrow[r, "{[0\,1]}"] \arrow[d, "1 \sqcup [h\,1]"'] & X \sqcup B \arrow[d, "{[h\,1]}"] \\ X \sqcup B \arrow[r, "{[0\,1]}"] & B \end{tikzcd}$$

is a pullback.

Specifically, given a morphism $h : X \longrightarrow B$ with trivial kernel (in pointed sets), the internal category it describes is the following: the objects are the elements of B; the morphisms are the identities 1_b for each $b \in B$ and also the elements $x \in X$, except for the distinguished element $0 \in X$ that is identified with $0 \in B$. The domain of every x in X is $0 \in B$ and the codomain is $h(x)$. Since all arrows (except identities) start from $0 \in B$, and because the kernel of h is trivial, two morphisms x and x' (other than 0) never compose. The picture is a star with all arrows from the origin with no nontrivial loops.

On the other hand, again in pointed sets, if considering the subclass of split epis that are product projections, that is, of the form

$$\begin{tikzcd} X \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "<0{,}1>"] \end{tikzcd} ;$$

then the following result is obtained.

A internal reflexive graph relative to product projections is given by a map

$$\xi : X \times B \longrightarrow B; \quad (x, b) \mapsto x \cdot b$$

such that $0 \cdot b = b$ for all $b \in B$; and it is of the form

$$\begin{tikzcd} X \times B \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\xi"'] & B \arrow[l, "<0{,}1>" description] \end{tikzcd} .$$

A internal precategory, relative to product projections, is given by

$$Y \times (X \times B) \xrightarrow{\mu} X \times B \xrightarrow{\xi} B \quad , \quad \begin{tikzcd} Y \arrow[r, shift left, "\alpha"] & X \arrow[l, shift left, "\beta"] \end{tikzcd}$$

such that

$$\begin{aligned} \alpha\beta &= 1, \\ \mu(y, x, b) &= (y +_b x,\ b), \\ 0 +_b x &= x = \beta(x) +_b 0 \\ (y +_b x) \cdot b &= \alpha(y) \cdot (x \cdot b); \end{aligned}$$

and it is of the form

$$\begin{tikzcd} Y\times(X\times B) \arrow[r, shift left=6, "\pi_2"] \arrow[r, shift left=2, leftarrow, "<0{,}1>"] \arrow[r, "\mu"] \arrow[r, shift right=2, leftarrow, "\beta\times<0{,}1>"'] \arrow[r, shift right=6, "\alpha\times\xi"'] & X\times B \arrow[r, shift left=2, "\pi_2"] \arrow[r, leftarrow, "<0{,}1>"] \arrow[r, shift right=2, "\xi"'] & B \end{tikzcd}.$$

In particular if $X=Y$ and $\alpha=\beta=1$, we obtain a internal category (not necessarily associative), because the square

$$\begin{tikzcd} X\times(X\times B) \arrow[r, "\pi_2"] \arrow[d, "1\times\xi"'] & X\times B \arrow[d, "\xi"] \\ X\times B \arrow[r, "\pi_2"] & B \end{tikzcd}$$

is a pullback.

This means that a internal category in pointed sets and relative to product projections is given by two maps

$$\begin{aligned} \mu \quad &: \quad X\times(X\times B)\longrightarrow X\times B;\quad (x,x',b)\mapsto(x+_b x',b) \\ \xi \quad &: \quad X\times B\longrightarrow B;\quad (x,b)\mapsto x\cdot b \end{aligned}$$

such that

$$\begin{aligned} 0+_b x &= x=x+_b 0 \\ 0\cdot b &= b \\ (x+_b x')\cdot b &= x\cdot(x'\cdot b) \end{aligned}$$

and in order to have associativity one must also require the additional condition

$$\left(x''+_{(x\cdot b)}x'\right)+_b x=x''+_b(x'+_b x)\,.$$

Specifically, given a structure as above in pointed sets, the corresponding internal category that it represents is the following. The objects are the elements of B. The morphisms are pairs (x,b) with domain b and codomain $x\cdot b$. The composition of

$$b\overset{(x,b)}{\longrightarrow}x\cdot b\overset{(x',b')}{\longrightarrow}x'\cdot(x\cdot b)$$

is the pair $(x'+_b x,b)$.

We will observe that for a given subclass of split epis, when the following two properties are present for every split epi (A,α,β,B) in the subclass:
(a) the morphism $\alpha:A\longrightarrow B$ has a kernel, say $k:X\longrightarrow A$
(b) the pair (k,β) is jointly epic
then a reflexive graph relative to the given subclass is determined by a split epi in the subclass, say (A,α,β,B) together with a *central morphism*

$$h:X\longrightarrow B$$

where a central morphism (see [**6**]) is such that there is a (necessarily unique) morphism, denoted by $[h\ 1]:A\longrightarrow B$ with the property

$$[h\ 1]\beta=1\quad,\quad[h\ 1]k=h,$$

where $k:X\longrightarrow A$ is the kernel of α.

In the case of Groups, considering the subclass of split epis given by semi-direct product projections

$$X \rtimes B \underset{<0,1>}{\overset{\pi_2}{\rightleftarrows}} B \ ,$$

the notion of central morphism $h : X \longrightarrow B$ (together with a semidirect product projection, or an internal group action) corresponds to the usual definition of pre-crossed module.

This fact may lead us to consider an abstract notion of semidirect product as a diagram in a category satisfying some universal property.

In [**7**] O. Berndt proposes the categorical definition of semidirect products as follows: the semidirect product of X and B (in a pointed category) is a diagram

$$X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

such that $\alpha\beta = 1$ and $k = \ker \alpha$.

We now see that it would be more reasonable to adjust this definition as follows: in a pointed category, the semidirect product of X and B, denoted $X \rtimes B$, is defined together with two morphisms

$$X \xrightarrow{k} X \rtimes B \xleftarrow{\beta} B$$

satisfying the following three conditions:
(a) the pair (k, β) is jointly epic
(b) the zero morphism

$$0 : X \longrightarrow B$$

is central, that is, there exists a (necessarily unique) morphism $[0\ 1] : X \rtimes B \longrightarrow B$ with $[0\ 1]\beta = 1$ and $[0\ 1]k = 0$
(c) k is the kernel of $[0\ 1]$.

We must add that this object $X \rtimes B$ may not be uniquely determined (even up to isomorphism), to achieve that we simply require the pair (k, β) to be universal with the above properties.

We also remark that we have not investigate further the consequences of such a definition. It will only be done in some future work. We choose to mention it at this point because it is related with the present work.

Another example, of considering internal categories relative to split epis, may be found in [**10**] where A. Patchkoria shows that, in the category of Monoids, the notion of internal category relative to semidirect product projections is in fact equivalent to the notion of a Schreier category.

This work is organized as follows.

First we recall some basic definitions, and introduce a concept that is obtained by weakening the notion of reflection, so that we choose to call it half-reflection.

Next we study the case of additivity, and find minimal conditions on a category $\mathbf{B}$ in order to have

$$\begin{array}{rcl} RG(\mathbf{B}) & \sim & Mor(\mathbf{B}) \\ PC(\mathbf{B}) & \sim & 2\text{-}Chains(\mathbf{B}) \end{array}$$

the usual equivalences between reflexive graphs and morphisms in $\mathbf{B}$, precategories and 2-chains in $\mathbf{B}$. We show that this is the case exactly when $\mathbf{B}$ is pointed (but not necessarily with a zero object), has binary coproducts and kernels of split epis,

and satisfies the following two conditions (see Theorem 9):
(a) ι_1 is the kernel of [01]

$$X \xrightarrow{\iota_1} X \sqcup B \xrightarrow{[0\ 1]} B$$

(b) the split short five lemma holds.

Later we investigate the same notions, and essentially obtain the same results, for the case of semi-additivity, by replacing coproduct projections by semidirect product projections, where the notion of semi-direct product is associated with the notion of internal actions in the sense of [**1**] and [**3**].

At the end we describe the same situation for a general setting, specially designed to mimic internal actions and semidirect products. An application of the results is given for the category of unitary magmas with right cancellation.

2. Definitions

Definition 1 (reflection). *A functor $I : \mathbf{A} \longrightarrow \mathbf{B}$ is a reflection when there is a functor*

$$H : \mathbf{B} \longrightarrow \mathbf{A}$$

and a natural transformation

$$\rho : 1_{\mathbf{A}} \longrightarrow HI$$

satisfying the following conditions

$$\begin{aligned} IH &= 1_{\mathbf{B}} \\ I \circ \rho &= 1_I \\ \rho \circ H &= 1_H . \end{aligned}$$

Definition 2 (half-reflection). *A pair of functors*

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \ , \quad , \quad IG = 1_{\mathbf{B}}$$

is said to be a half-reflection if there is a natural transformation

$$\pi : 1_{\mathbf{A}} \longrightarrow GI$$

such that

$$I \circ \pi = 1_I .$$

Theorem 1. *For a half-reflection (I, G, π) we always have*

(2.1)
$$\begin{tikzcd} 1 \arrow[r, "\pi"] \arrow[dr, "\pi"'] & GI \arrow[d, "\pi \circ GI"] \\ & GI \end{tikzcd} .$$

Proof. By naturality of π we have

$$\begin{tikzcd} A \arrow[r, "\pi_A"] \arrow[d, "\pi_A"'] & GIA \arrow[d, "GI(\pi_A)"] \\ GIA \arrow[r, "\pi_{GIA}"] & GI(GIA) \end{tikzcd}$$

but $GI(GIA) = GIA$ and $GI\,(\pi_A) = 1_{GIA}$. $\square$

When it exists, the natural transformation $\pi : 1 \longrightarrow GI$ is *essentially* unique, in the sense that any other such, say $\pi' : 1 \longrightarrow GI$ (with $I \circ \pi' = 1_I$), is of the form

$$\pi'_A = \pi'_{GIA}\pi_A.$$

Under which conditions is it really unique?

The name half-reflection is motivated because if instead of (2.1) we have $\pi \circ G = 1_G$ then the result is a reflection.

For any category $\mathbf{B}$ we consider the category of internal reflexive graphs in $\mathbf{B}$, denoted $RG(\mathbf{B})$ as usual:
Objects are diagrams in $\mathbf{B}$ of the form

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce;$$

Morphisms are pairs (f_1, f_0) making the obvious squares commutative in the following diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[l, "e" description] \arrow[d, "f_0"] \\ C'_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C'_0 \arrow[l, "e'" description] \end{tikzcd} .$$

We will also consider the category of internal precategories in $\mathbf{B}$, denoted $PC(\mathbf{B})$, where objects are diagrams of the form

$$\begin{tikzcd} C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2" description] \arrow[l, shift left=2, "e_1" description] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that

(2.2)
$$\begin{tikzcd} C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\ C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd}$$

is a split square (i.e. a split epi in the category of split epis), so that in particular we have

(2.3)
$$de = 1_{C_0} = ce$$

and furthermore, the following three conditions are satisfied

(2.4)
$$dm = d\pi_2$$

(2.5)
$$cm = c\pi_1.$$

(2.6)
$$me_1 = 1_{C_1} = me_2;$$

and obvious morphisms.

A precategory in this sense becomes a category[1] if the top and left square in (2.2) is a pullback.

Definition 3. *A category is said to have coequalizers of reflexive graphs if for every reflexive graph*

$$C_1 \underset{c}{\overset{d}{\substack{\longrightarrow \\ \xleftarrow{e} \\ \longrightarrow}}} C_0 \quad , \quad de = 1 = ce$$

the coequalizer of d and c exists.

Definition 4 (pointed category). *A pointed category is a category enriched in pointed sets. More specifically, for every pair X, Y of objects, there is a specified morphism, $0_{X,Y} : X \longrightarrow Y$ with the following property:*

$$X \overset{0_{X,Y}}{\longrightarrow} Y \overset{f}{\longrightarrow} Z \overset{0_{Z,W}}{\longrightarrow} W$$

$$0_{Z,W} f = 0_{Y,W} \tag{2.7}$$

$$f 0_{X,Y} = 0_{X,Z}. \tag{2.8}$$

Definition 5 (additive category). *An additive category is an Ab-category with binary biproducts.*

Observe that on the contrary to the usual practice we are not considering the existence of a null object, neither in pointed nor in additive categories.

3. Additivity

Let $\mathbf{B}$ be any category and consider the pair of functors

$$\mathbf{B} \times \mathbf{B} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}$$

with $I(X,B) = B$ and $G(B) = (B,B)$.

Theorem 2. *The above pair (I,G) is a half-reflection if and only if the category $\mathbf{B}$ is pointed.*

Proof. If $\mathbf{B}$ is pointed simply define

$$\pi_{(X,B)} : (X,B) \longrightarrow (B,B)$$

as $\pi_{(X,B)} = (0_{X,B}, 1_B)$.
Now suppose there is a natural transformation

$$\pi : 1_{\mathbf{BxB}} \longrightarrow GI,$$

such that $I \circ \pi = 1_I$, this is the same as having for every pair X, B in $\mathbf{B}$ a specified morphism

$$\pi_{X,B} : X \longrightarrow B$$

[1] In fact it is not quite a category because we are not considering associativity; also the term precategory is often used when (2.6) is not present; we also observe that in many interesting cases (for example in Mal'cev categories) assuming only (2.6) and the fact that (2.2) is a pullback, then the resulting structure is already an internal category.

and conditions (2.7) and (2.8) follow by naturality:

$$\begin{tikzcd}
(Y,W) \arrow[r, "{(\pi_{Y,W},1)}"] \arrow[d, "{(f,1)}"'] & (W,W) \arrow[d, "{(1,1)}"] \\
(Z,W) \arrow[r, "{(\pi_{Z,W},1)}"] & (W,W)
\end{tikzcd} \quad , \qquad
\begin{tikzcd}
(X,Y) \arrow[r, "{(\pi_{X,Y},1)}"] \arrow[d, "{(1,f)}"'] & (Y,Y) \arrow[d, "{(f,f)}"] \\
(X,Z) \arrow[r, "{(\pi_{X,Z},1)}"] & (Z,Z)
\end{tikzcd} \quad .$$

□

THEOREM 3. *The functor G as above admits a left adjoint if and only if the category $\mathbf{B}$ has binary coproducts.*

PROOF. As it is well known, coproducts are obtained as the left adjoint to the diagonal functor. □

THEOREM 4. *Given a half-reflection (I, G, π)*

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \quad , \qquad \pi : 1_{\mathbf{A}} \longrightarrow GI,$$

if the functor G admits a left adjoint

$$(F, G, \eta, \varepsilon)\,,$$

then, there is a canonical functor

$$\mathbf{A} \longrightarrow Pt\,(\mathbf{B})$$

sending an object $A \in \mathbf{A}$ to the split epi

$$FA \underset{I(\eta_A)}{\overset{\varepsilon_{IA}F(\pi_A)}{\rightleftarrows}} IA\ .$$

PROOF. We only have to prove

$$\varepsilon_{IA} F\,(\pi_A)\, I\,(\eta_A) = 1_{IA}.$$

Start with

$$\pi_A = G\,(\varepsilon_{IA} F\,(\pi_A))\,\eta_A$$

and apply I to both sides to obtain

$$I\,(\pi_A) = \varepsilon_{IA} F\,(\pi_A)\, I\,(\eta_A)\,,$$

by definition we have $I\,(\pi_A) = 1_{IA}$. □

In particular if $\mathbf{B}$ is pointed and has binary coproducts we have the canonical functor

$$\mathbf{B} \times \mathbf{B} \overset{T}{\longrightarrow} Pt\,(\mathbf{B})$$

sending a pair (X, B) to the split epi

$$X \sqcup B \underset{\iota_2}{\overset{[0\ 1]}{\rightleftarrows}} B\ .$$

THEOREM 5. *Let* $\mathbf{B}$ *be a pointed category with binary coproducts. The canonical functor* $\mathbf{B}\times\mathbf{B} \xrightarrow{T} Pt\,(\mathbf{B})$ *admits a right adjoint,* S*, such that* $IS = I'$

$$\begin{tikzcd} \mathbf{B}\times\mathbf{B} \arrow[dr, "I"'] & & Pt(\mathbf{B}) \arrow[ll, "S"'] \arrow[dl, "I'"] \\ & \mathbf{B} & \end{tikzcd}$$

if and only if the category $\mathbf{B}$ *has kernels of split epis.*
The functor I' *sends a split epi* (A,α,β,B) *to* B*.*

PROOF. If the category has kernels of split epis, then for every split epi we choose a specified kernel

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and the functor S, sending (A,α,β,B) to the pair (X,B) is the right adjoint for T:

$$\begin{tikzcd} (Y,D) \arrow[d, "{(f,g)}"] \\ (X,B) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\iota_1"] \arrow[d, "f"'] & Y\sqcup D \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[kf\;\beta g]}"] & B \arrow[l, shift left, "\iota_2"] \arrow[d, "g"] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} .$$

Now suppose S is a right adjoint to T and it is such that a split epi (A,α,β,B) goes to a pair of the form

$$(K[\alpha],B)$$

with unit and counit as follows

$$\begin{tikzcd} (X,B) \arrow[d, "{(\eta_X,1)}"] \\ (K[0\,1],B) \end{tikzcd} \qquad \begin{tikzcd} K[\alpha] \arrow[r, "\iota_1"] \arrow[dr, "\varepsilon_1"'] & K[\alpha]\sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[\varepsilon_1\;\beta]}"] & B \arrow[l, shift left, "\iota_2"] \arrow[d, "1"] \\ & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} .$$

We have to prove that $\varepsilon_1 = \ker\alpha$, and in fact, we have $\alpha\varepsilon_1 = 0$ and by the universal property of the counit we have that given a morphism of split epis

$$\begin{tikzcd} X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X\sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[f\;\beta]}"] & B \arrow[l, shift left, "\iota_2"] \arrow[d, "1"] \\ & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

that is a morphism $f : X \longrightarrow A$ such that $\alpha f = 0$, there exists a unique $(f',1) : (X,B) \longrightarrow (K[\alpha],B)$ such that

$$[\varepsilon_1\;\beta]\,(f'\sqcup 1) = [f\;\beta]$$

which is equivalent to say $\varepsilon_1 f' = f$. Hence ε_1 is a kernel for α. $\square$

THEOREM 6. *Let* $\mathbf{B}$ *be a pointed category with binary coproducts and kernels of split epis. If the canonical adjunction*

$$\begin{tikzcd} \mathbf{B}\times\mathbf{B} \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & Pt\left(\mathbf{B}\right) \arrow[l, shift left, "S"] \end{tikzcd}$$

is an equivalence, then:

$$RG\left(\mathbf{B}\right)\sim Mor\left(\mathbf{B}\right)$$

and

$$PC\left(\mathbf{B}\right)\sim 2\text{-}Chains\left(\mathbf{B}\right).$$

PROOF. By the equivalence we have that a split epi

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \end{tikzcd} \qquad , \quad de=1$$

is of the form

$$\begin{tikzcd} X\sqcup B \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd} ,$$

and to give a morphism $c: X\sqcup B\longrightarrow B$ such that $c\iota_2=1$ is to give a morphism

$$h: X\longrightarrow B.$$

So that a reflexive graph is, up to isomorphism, of the form

$$\begin{tikzcd} X\sqcup B \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=2, "{[h\,1]}"'] & B \arrow[l, "\iota_2" description] \end{tikzcd} .$$

To investigate a precategory we observe that the square (2.2) may be considered as a split epi in the category of split epis, and hence, it is given up to an isomorphism in the form

$$\begin{tikzcd} Y\sqcup\left(X\sqcup B\right) \arrow[r, shift left, "{[0\,1]}"] \arrow[d, shift right, "{a\sqcup[h,1]}"'] & X\sqcup B \arrow[l, shift left, "\iota_2"] \arrow[d, shift right, "{[h\,1]}"'] \\ X\sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[u, shift right, "b\sqcup\iota_2"'] & B \arrow[l, shift left, "\iota_2"] \arrow[u, shift right, "\iota_2"'] \end{tikzcd} \qquad , \quad ab=1. \tag{3.1}$$

It follows that m, satisfying $m\iota_2=1$ is of the form

$$Y\sqcup\left(X\sqcup B\right)\overset{[v\ 1]}{\longrightarrow}\left(X\sqcup B\right)$$

and hence to give m is to give $v: Y\longrightarrow X\sqcup B$.
Since we also have (2.4) then $[0\ 1]v=0$, and v factors through the kernel of $[0\ 1]$ which is (see Theorem 9)

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & X\sqcup B \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd} .$$

This shows that to give m is to give a morphism

$$u: Y\longrightarrow X$$

and hence m is given as

$$m=\left[\iota_1 u\ 1\right]: Y\sqcup\left(X\sqcup B\right)\longrightarrow\left(X\sqcup B\right).$$

Finally we have that condition (2.5) is equivalent to $ha = hu$ and $m(b \sqcup \iota_2)$ is equivalent to $ub = 1$.
Conclusion 1: A precategory in $\mathbf{B}$ is completely determined by a diagram

$$\begin{tikzcd} Y \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "u"'] & X \arrow[l, "b"'] \arrow[r, "h"] & B \end{tikzcd}$$

such that

$$ab = 1 = ub\ ,\ ha = hu.$$

Continuing with a further analysis we observe that the resulting diagram is in particular a reflexive graph and hence it is, up to isomorphism, of the form

$$\begin{tikzcd} Z \sqcup X \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=2, "{[t\,1]}"'] & X \arrow[l, "\iota_2"'] \arrow[r, "h"] & B \end{tikzcd}$$

where $h[0\ 1] = h[t\ 1]$ is equivalent to $ht = 0$.
Conclusion 2: A precategory in $\mathbf{B}$ is completely determined by a 2-chain complex

$$Y \xrightarrow{t} X \xrightarrow{h} B\quad ,\quad ht = 0.$$

□

Remark 1. *In the future we will not assume the canonical functor T to be an equivalence, and hence the second conclusion will no longer be possible. However, we will be interested in the study of precategories such that* (2.2) *is of the form* (3.1) *and in that case, provided that ι_1 is the kernel of* $[0\ 1]$ *we still can deduce conclusion 1. Such an example is the category of pointed sets: see Introduction.*

There is a canonical inclusion of reflexive graphs into precategories, by sending $h : X \longrightarrow B$ to

$$\begin{tikzcd} X \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & X \arrow[l, "1"'] \arrow[r, "h"] & B\ . \end{tikzcd}$$

Theorem 7. *If $\mathbf{B}$ has coequalizers of reflexive graphs, then the canonical functor*

$$PC(\mathbf{B}) \underset{V}{\longleftarrow} RG(\mathbf{B})$$

has a left adjoint.

Proof. The left adjoint is the following.
Given the precategory

$$\begin{tikzcd} Y \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "u"'] & X \arrow[l, "b"'] \arrow[r, "h"] \arrow[dr, "\sigma = coeq"'] & B \\ & & X' \arrow[u, dashed, "h'"'] \end{tikzcd}$$

construct the coequalizer of u and a, say σ, and consider the reflexive graph in $\mathbf{B}$ determined by

$$h' : X' \longrightarrow B.$$

This defines a reflection

$$PC(\mathbf{B}) \xrightarrow{U} RG(\mathbf{B})$$

with unit

$$\begin{tikzcd}
Y \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "u"'] \arrow[d, "\sigma u = \sigma a"'] & X \arrow[l, "b" description] \arrow[r, "h"] \arrow[d, "\sigma"'] & B \arrow[d, equal] \\
X' \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & X' \arrow[l, "1" description] \arrow[r, "h'"] & B
\end{tikzcd} .$$

□

Next we characterize a category $\mathbf{B}$, pointed, with binary coproducts and such that the canonical functor

$$\mathbf{B} \times \mathbf{B} \longrightarrow Pt\,(\mathbf{B})$$

is an equivalence.

First observe that:

Theorem 8. *If $\mathbf{B}$, as above, also has binary products, then it is an additive category (with kernels of split epis).*

Proof. We simply observe that in particular

$$\begin{tikzcd}
X \sqcup B \arrow[r, shift left, "{[0\ 1]}"] \arrow[d, "\cong"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\
X \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\langle 0,1 \rangle"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X \sqcup B \arrow[r, shift left, "{[0\ 1]}"] \arrow[d, "\cong"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\
X \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\langle 1,1 \rangle"]
\end{tikzcd} ,$$

since $X \overset{\langle 1,0 \rangle}{\longrightarrow} X \times B$ is a kernel for π_2. See [**2**] for more details. □

Theorem 9. *Let $\mathbf{B}$ be pointed with binary coproducts and kernels of split epis. The following conditions are equivalent:*

(a): *the canonical adjunction*

$$\begin{tikzcd}
\mathbf{B} \times \mathbf{B} \arrow[r, shift left=2, "T"] \arrow[r, phantom, "\perp"] & Pt\,(\mathbf{B}) \arrow[l, shift left=2, "S"]
\end{tikzcd}$$

$$\begin{array}{rcl} T\,(X, B) & = & (X \sqcup B, [0\ 1], \iota_2, B) \\ S\,(A, \alpha, \beta, B) & = & (K[\alpha], B)\,, \end{array}$$

is an equivalence of categories;

(b): *the category $\mathbf{B}$ satisfies the following two axioms:*

(A1): *for every diagram of the form*

(3.2) $$X \xrightarrow{\iota_1} X \sqcup B \underset{\iota_2}{\overset{[0\ 1]}{\rightleftarrows}} B$$

the morphism ι_1 is the kernel of $[0\ 1]$;

(A2): *the split short five lemma holds, that is, given any diagram of split epis and respective kernels*

(3.3) $$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "f"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "h"'] & B \arrow[l, shift left, "\beta"] \arrow[d, "g"] \\
X' \arrow[r, "k'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"]
\end{tikzcd}$$

if g and f are isomorphisms then h is an isomorphism.

Proof. $(b) \Rightarrow (a)$ Using only (A1) we have that $ST \cong 1$, and using (A1) and (A2) we have, in particular, that $[k\ \beta]$ as in

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[d, equal] & X \sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[k\ \beta]}"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\
X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

is an isomorphism, and hence $TS \cong 1$.

$(a) \Rightarrow (b)$ Suppose $ST \cong 1$, this gives (A1); suppose $TS \cong 1$, so that from (3.3) we can form

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[d, "f"'] & X \sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "f \sqcup g"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, "g"] \\
X' \arrow[r, "\iota_1"] & X \sqcup B \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "\iota_2"]
\end{tikzcd}$$

and if f, g are isomorphisms, we can find $h^{-1} = [k\ \beta] \left(f^{-1} \sqcup g^{-1}\right) [k'\ \beta']^{-1}$. □

Corollary 1. *If T is a reflection then it is an equivalence of categories.*

We may now state the following results.

Conclusion 1. *Let $\mathbf{B}$ be a pointed category with binary coproducts. TFAE:*
(a) T is a reflection and $\mathbf{B}$ has binary products;
(b) $\mathbf{B}$ is additive and has kernels of split epis.

Conclusion 2. *Let $\mathbf{B}$ be pointed, with binary products and coproducts and kernels of split epis. TFAE:*
(a) T is a reflection;
(b) $\mathbf{B}$ is additive.

3.1. Restriction to split epis. Suppose now that the canonical functor T is not an equivalence, but we still have axiom (3.2), that is $ST \cong 1$. The results relating precategories and reflexive graphs will still hold if we restrict $PC(\mathbf{B})$ to diagrams of the form

$$\begin{tikzcd}
Y \sqcup (X \sqcup B) \arrow[r, shift left=4.5, "{[0\,1]}"] \arrow[r, "m"] \arrow[r, shift right=4.5, "a \sqcup c"'] & X \sqcup B \arrow[l, shift right=2.25, "\iota_2"'] \arrow[l, shift left=2.25, "b \sqcup c"] \arrow[r, shift left=3, "{[0\,1]}"] \arrow[r, shift right=3, "c"'] & B \arrow[l, "\iota_2"']
\end{tikzcd} .$$

This result will be proved in a more general case in the next sections.

An example of such a case is the category of pointed sets.

If starting with a general half-reflection

$$\begin{tikzcd}
\mathbf{A} \arrow[r, shift left, "I"] & \mathbf{B} \arrow[l, shift left, "G"]
\end{tikzcd} \quad , \quad \pi : 1 \longrightarrow GI$$

such that G admits a left adjoint

$$(F, G, \eta, \varepsilon)$$

we consider the canonical functor

$$\mathbf{A} \xrightarrow{T} Pt(\mathbf{B})$$

and ask if it is an equivalence; if not we then ask if it satisfies at least one of the axioms (3.2) or (3.3). For example for $\mathbf{A} = \mathbf{B} \times \mathbf{B}$ and assuming the constructions as above, in the case of pointed sets we have (3.2) but not (3.3), while in groups we have (3.3) but not (3.2).

In the case where we have only (3.2) we will be interested in the study of $RG(\mathbf{B})$ and $PC(\mathbf{B})$ restricted to split epis of the form

$$FA \underset{I(\eta_A)}{\overset{\varepsilon_{IA}F(\pi_A)}{\rightleftarrows}} IA\ ,$$

while if in the presence of (3.3), but not (3.2), we may construct a category of internal actions as suggested in [**1**].

4. Semi-Additivity

Let $\mathbf{B}$ be a pointed category with binary coproducts and kernels of split epis. As shown in the previous section there is a canonical adjunction

$$\mathbf{B} \times \mathbf{B} \underset{S}{\overset{T}{\underset{\perp}{\rightleftarrows}}} Pt(\mathbf{B})\,. \tag{4.1}$$

We are considering $\mathbf{B} \times \mathbf{B}$ and $Pt(\mathbf{B})$ as objects in the category of functors over $\mathbf{B}$, that is

$$\begin{tikzcd} \mathbf{B} \times \mathbf{B} \arrow[dr, "I"'] & & Pt(\mathbf{B}) \arrow[dl, "I'"] \\ & \mathbf{B} & \end{tikzcd}$$

where $I(X, B) = B$ and $I'(A, \alpha, \beta, B) = B$.

We are also interested in the fact that I is a half-reflection, with respect to some functor G. In the case of $\mathbf{B} \times \mathbf{B}$ there is a canonical choice for G, namely the diagonal functor, and it is a half-reflection if and only if $\mathbf{B}$ is pointed. We are also interested in the fact that G admits a left adjoint.

In the case of $Pt(\mathbf{B})$ there are apparently many good choices for the functor G' to be a half-reflection together with I'. Nevertheless, if we ask that the left adjoint for G' to be F', such that $F'(A, \alpha, \beta, B) = A$, then we calculate G' as follows.

Theorem 10. *Let* $\mathbf{B}$ *be any category and consider the two functors*

$$Pt(\mathbf{B}) \underset{F}{\overset{I}{\rightrightarrows}} \mathbf{B}$$

$$\begin{aligned} I(A, \alpha, \beta, B) &= B \\ F(A, \alpha, \beta, B) &= A. \end{aligned}$$

The functor F *admits a right adjoint*

$$(F, G, \eta, \varepsilon)$$

such that $IG = 1_{\mathbf{B}}$ *if and only if the category* $\mathbf{B}$ *has an endofunctor*

$$G_1 : \mathbf{B} \longrightarrow \mathbf{B}$$

and natural transformations

$$\begin{tikzcd} G_1(B) \arrow[r, shift left=2, "\pi_B"] \arrow[r, shift right=2, "\varepsilon_B"'] & B \arrow[l, "\delta_B" description] \end{tikzcd} \quad , \qquad \pi_B \delta_B = 1_B,$$

satisfying the following property:
for every diagram in $\mathbf{B}$ *of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[dr, "f"'] & B' \arrow[l, shift left, "\beta"] \\ & B \end{tikzcd} \quad , \quad \alpha\beta = 1$$

there exists a unique morphism

$$f' : A \longrightarrow G_1(B)$$

such that

$$\begin{aligned} \varepsilon_B f' &= f \\ \delta_B \pi_B f' &= f' \beta \alpha. \end{aligned}$$

Proof. Suppose we have $G_1, \pi, \delta, \varepsilon$ satisfying the required conditions in the Theorem, then the functor

$$G(B) = \begin{tikzcd} G_1(B) \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\delta_B"] \end{tikzcd}$$

is a right adjoint to F; in fact (see [**4**], p.83, Theorem 2, (iii)) we have functors F and G, and a natural transformation $\varepsilon : FG \longrightarrow 1$, such that each $\varepsilon_B : FG(B) \longrightarrow B$ is universal from F to B:

$$\begin{tikzcd} A \arrow[d, "f"] \\ B' \end{tikzcd} \quad : \quad \begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[d, "f_1"] \arrow[dr, dashed, "f"] & B \arrow[l, shift left, "\beta"] \arrow[d, "f_0"] \\ G_1(B') \arrow[r, shift left, "\pi_B"] & B' \arrow[l, shift left, "\delta_B"] \end{tikzcd}$$

given f, there is a unique f_1 (with $\varepsilon_B f' = f$, $\delta_B \pi_B f' = f' \beta \alpha$) and f_0 follows as $f_0 = \pi_B f_1 \beta$; conversely, given f_1, we find $f = \varepsilon_B f_1$.

Now, given an adjunction

$$(F, G, \eta, \varepsilon)$$

such that $IG = 1$, if writing

$$G(B) = \begin{tikzcd} G_1(B) \arrow[r, shift left, "G_2(B)"] & B \arrow[l, shift left, "G_3(B)"] \end{tikzcd}$$

we define

$$G_1 = FG \quad , \quad \pi_B = G_2(B) \quad , \quad \delta_B = G_3(B)$$

and

$$\varepsilon_B : FG(B) \longrightarrow B$$

is the counit of the adjunction.

Clearly we have natural transformations with $\pi_B \delta_B = 1$.

It remains to check the stated property - but it is simply the universal property of ε_B: given a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[dr, "f"'] & B' \arrow[l, shift left, "\beta"] \\ & B \end{tikzcd} \quad , \quad \alpha\beta = 1$$

there is a unique morphism of split epis

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[d, "f_1"] & B \arrow[l, shift left, "\beta"] \arrow[d, "f_0"] \\ G_1(B') \arrow[r, shift left, "\pi_B"] & B' \arrow[l, shift left, "\delta_B"] \end{tikzcd}$$

such that $\varepsilon_B f_1 = f$; being a morphism of split epis means that $f_0 = \pi_B f_1 \beta$, and f_1 is such that $\delta_B \pi_B f_1 = f_1 \beta\alpha$. $\square$

If $\mathbf{B}$ has binary products, then for every $B \in \mathbf{B}$,

$$\begin{tikzcd} B \times B \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & B \arrow[l, "<1,1>" description] \end{tikzcd}$$

satisfies the required conditions and hence F has a right adjoint, G, sending the object B to the split epi

$$\begin{tikzcd} B \times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "<1,1>"] \end{tikzcd} .$$

And furthermore, in this case the pair (I, G) is a half-reflection. For the general case, if we ask for (I, G) to be a half-reflection, then the following result suffices.

Corollary 2. *Let $\mathbf{B}$ be any category and $I, F : Pt(\mathbf{B}) \longrightarrow \mathbf{B}$ as above. If the category $\mathbf{B}$ is equipped with an endofunctor $G_1 : \mathbf{B} \longrightarrow \mathbf{B}$ and natural transformations*

$$\begin{tikzcd} G_1(B) \arrow[r, shift left=2, "\pi_B"] \arrow[r, shift right=2, "\varepsilon_B"'] & B \arrow[l, "\delta_B" description] \end{tikzcd} \quad , \quad \pi_B \delta_B = 1_B = \varepsilon_B \delta_B,$$

satisfying the following property:
for every diagram in $\mathbf{B}$ of the form

$$\begin{tikzcd} A \arrow[dr, "f"'] & A \arrow[l, "t"] \\ & B \end{tikzcd} \quad , \quad t^2 = t$$

there exists a unique morphism

$$f' : A \longrightarrow G_1(B)$$

such that

$$\pi_B f' = ft \ , \ \varepsilon_B f' = f \ , \ f't = \delta_B ft,$$

then the functor F has a right adjoint, say G, and the pair (I, G) is a half reflection.

Proof. It is clear that the property is sufficient to obtain G as a right adjoint to F as in the previous Theorem, simply considering $t = \beta\alpha$ and observing that the two conditions $\pi_B f' = ft$, $f't = \delta_B ft$ give $\delta_B \pi_B f' = f'\beta\alpha$: start with $\pi_B f' = ft$, precompose with δ_B, and replace $\delta_B ft$ by $f't$.

As a consequence we have that

$$\pi_B f'\beta = \varepsilon_B f'\beta,$$

since

$$\begin{aligned} ft = ftt = \pi_B f't = \pi_B f'\beta\alpha = \pi_B f'\beta \\ ft = \varepsilon_B \delta_B ft = \varepsilon_B f't = \varepsilon_B f'\beta\alpha = \varepsilon_B f'\beta \end{aligned}$$

and hence, given $f : A \longrightarrow B$, we have (f_1, f_0), with $f_1 = f'$ given by the *universal* property and $f_0 = f\beta$ $(= \pi_B f'\beta = \varepsilon_B f'\beta)$.

The pair (I, G) is a half-reflection with

$$\pi : 1_{Pt(\mathbf{B})} \longrightarrow GI$$

given by

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[d, dashed, "\delta_B \alpha"] & B \arrow[l, shift left, "\beta"] \arrow[d, equal] \\ G_1(B) \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\delta_B"] \end{tikzcd} ,$$

and furthermore this is the only possibility. $\square$

Corollary 3. *In the conditions of the above Corollary (and assuming* $\mathbf{B}$ *is pointed), the kernel of* π_B *is the morphism induced by the diagram*

$$\begin{tikzcd} B \arrow[dr, "1"'] & B \arrow[l, "0"] \\ & B \end{tikzcd} .$$

As mentioned in the previous section, if the canonical adjunction (4.1) is not an equivalence we are interested in considering bigger categories, $\mathbf{A}$, that we will call categories of actions, in the place of $\mathbf{B} \times \mathbf{B}$, in order to obtain an equivalence of categories $\mathbf{A} \sim Pt(\mathbf{B})$.

We now turn our attention to the category of internal actions in $\mathbf{B}$.

To define the category of internal actions in $\mathbf{B}$, in the sense of [**1**], we only need to assume $\mathbf{B}$ to be pointed, with binary coproducts and kernels of split epis: exactly the same conditions necessary to consider the canonical adjunction (4.1); and the construction of the category of internal actions is actually suggested by the adjunction. This seems to suggest an iterative process to obtain bigger and bigger categories "of actions", $\mathbf{A}_1$, $\mathbf{A}_2$,....

4.1. The category of internal actions. Let $\mathbf{B}$ be a pointed category with binary coproducts and kernels of split epis. The category of internal actions in $\mathbf{B}$, denoted $Act(\mathbf{B})$, is defined as follows.

Objects are triples (X,ξ,B) where X and B are objects in $\mathbf{B}$ and $\xi : B\flat X \longrightarrow X$ is a morphism such that

$$\begin{aligned} \xi\eta_X &= 1 \\ \xi\mu_X &= \xi\,(1\flat\xi) \end{aligned}$$

where the object $B\flat X$ is the kernel, $k : B\flat X \longrightarrow X \sqcup B$ of $[0,1] : X \sqcup B \longrightarrow B$ and η_X, μ_x are induced, respectively, by ι_1 and $[k\ \iota_2]$. See [**1**] for more details.

We now have to consider $Act\,(\mathbf{B})$ as an half-reflection, (I,G) (with G admitting a left adjoint), over $\mathbf{B}$. Clearly we have a functor

$$\begin{tikzcd} \mathbf{A} \arrow[dr, "I"'] & & Pt(\mathbf{B}) \arrow[ll, "S"'] \arrow[dl, "I'"] \\ & \mathbf{B} & \end{tikzcd}$$

sending a split epi (A,α,β,B) to (X,ξ,B) as suggested in the following diagram

$$\begin{tikzcd} B\flat X \arrow[r, "k'"] \arrow[d, dashed, "\xi"'] & X \sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[k\,\beta]}"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\ X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} .$$

It is well defined because k (being a kernel) is monic and

$$k\xi\eta_X = [k\ \beta]k'\eta_X = [k\ \beta]\iota_1 = k$$

so that $\xi\eta_X = 1$; a similar argument shows $\xi\mu_X = \xi\,(1\flat\xi)$.

To obtain a functor $G : \mathbf{B} \longrightarrow Act\,(\mathbf{B})$ we compose

$$\mathbf{B} \longrightarrow Pt\,(\mathbf{B}) \longrightarrow Act\,(\mathbf{B})$$

where $\mathbf{B} \longrightarrow Pt\,(\mathbf{B})$ is the half-reflection of Corollary 2; the resulting G sends an object $B \in \mathbf{B}$ to the internal action $(B'\flat B, \xi_B, B)$ as suggested by the following diagram

$$\begin{tikzcd} B'\flat B \arrow[r, "\ker"] \arrow[d, dashed, "\xi_B"'] & B' \sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[\ker\ \delta_B]}"'] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\ B' \arrow[r, "\ker"] & G_1(B) \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\delta_B"] \end{tikzcd} .$$

In the case of Groups this corresponds to the action by conjugation (see [**1**]). The next step is to require that G admits a left adjoint, which in the case of Groups is true and it corresponds to the construction of a semi-direct product from a given action.

For convenience, we will now assume the existence of binary products, instead of the data $G_1, \pi, \delta, \varepsilon$ of Theorem 10.

For the rest of this section, and if not explicitly stated otherwise, we will assume that $\mathbf{B}$ is a pointed category with binary products and coproducts and kernels of split epis.

With such assumptions we automatically consider the half-reflection

$$\begin{tikzcd} Act\,(\mathbf{B}) \arrow[r, shift left, "I"] & \mathbf{B} \arrow[l, shift left, "G"] \end{tikzcd} \quad , \quad \pi : 1 \longrightarrow GI$$

where $I\,(X,\xi,B) = B$, $G\,(B) = (B,\xi_B,B)$ obtained from

$$\begin{tikzcd} B\flat B \arrow[r, "k"] \arrow[d, dashed, "\xi_B"'] & B\sqcup B \arrow[r, shift left, "{[0\,1]}"] \arrow[d, "{[<1,0>\,<1,1>]}"] & B \arrow[l, shift left, "\iota_2"] \arrow[d, equal] \\ B \arrow[r, "<1,0>"] & B\times B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "<1,1>"] \end{tikzcd}$$

(note that $<0,1>$ is the kernel of π_2), and the natural transformation $\pi : 1 \longrightarrow GI$ given by

$$\begin{tikzcd} (X,\xi,B) \arrow[d, "{(0,1)}"'] \\ (B,\xi_B,B) \end{tikzcd} \qquad \begin{tikzcd} B\flat X \arrow[r, "\xi"] \arrow[d, "1\flat 0"'] & X \arrow[d, "0"] \\ B\flat B \arrow[r, "\xi_B"] & B \end{tikzcd} \; ;$$

which is well defined because $\xi_B\,(1\flat 0) = 0$, since $\langle 1,0\rangle\,\xi_B\,(1\flat 0) = \langle 0,0\rangle$:

$$\langle 1,0\rangle\,\xi_B\,(1\flat 0) = [\langle 1,0\rangle \;\; \langle 1,1\rangle]\,(0\sqcup 1)\ker[0\;1] =$$
$$= \langle [1\;1],[0\;1]\rangle\,(0\sqcup 1)\ker[0\;1] = \langle 0,[0\;1]\rangle\ker[0\;1] = \langle 0,0\rangle\,.$$

Definition 6 (semi-direct products). *We will say that* $\mathbf{B}$ *has semidirect products, if the functor* G *admits a left adjoint.*

Note that this is a weaker notion of Bourn-Janelidze categorical semidirect products [**3**], since we are not asking for the induced adjuntion between $Act(\mathbf{B})$ and $Pt(\mathbf{B})$ to be an equivalence of categories.

We now state a sufficient condition for $\mathbf{B}$ to have semidirect products.

Theorem 11. *The functor* G*, in the half-reflection*

$$\begin{tikzcd} Act\,(\mathbf{B}) \arrow[r, shift left, "I"] & \mathbf{B} \arrow[l, shift left, "G"] \end{tikzcd} \quad , \quad \pi : 1 \longrightarrow GI,$$

as above, admits a left adjoint if the category $\mathbf{B}$ *has coequalizers of reflexive graphs.*

Proof. Given an object (X,ξ,B), consider the reflexive graph

$$\begin{tikzcd} (B\flat X)\sqcup B \arrow[r, shift left=2, "{[k\;\iota_2]}"] \arrow[r, shift right=2, "\xi\sqcup 1"'] & X\sqcup B \arrow[l, "\eta\sqcup 1" description] \end{tikzcd}\,.$$

The left adjoint, F, is given by the coequalizer of $[k\;\iota_2]$ and $\xi\sqcup 1$:

$$\begin{tikzcd} (B\flat X)\sqcup B \arrow[r, shift left=2, "{[k\;\iota_2]}"] \arrow[r, shift right=2, "\xi\sqcup 1"'] & X\sqcup B \arrow[l, "\eta\sqcup 1" description] \arrow[r, "\sigma"] & F(X,\xi,B) \end{tikzcd}\,.$$

See [**1**] for more details. □

Let us from now on assume that $\mathbf{B}$ is a pointed category with binary products and coproducts and kernels of split epis, and coequalizers of reflexive graphs.

The next step is to consider the canonical functor

$$Act\,(\mathbf{B}) \xrightarrow{T} Pt\,(\mathbf{B})$$

sending (X, ξ, B) to the split epi $\left(F\,(X, \xi, B)\,, \overline{[0\ 1]}, \sigma\iota_2\right)$ where $\overline{[0\ 1]}$ is such that $\overline{[0\ 1]}\sigma = [0\ 1]$, investigate whether it is an equivalence of categories and study internal precategories and reflexive graphs in $\mathbf{B}$.

First we show that under the given assumptions, it is always an adjunction.

Theorem 12. *The functors*

$$\begin{tikzcd} Act\,(\mathbf{B}) \arrow[r, shift left, "T"] & Pt\,(\mathbf{B}) \arrow[l, shift left, "S"] \end{tikzcd}$$

as defined above, form an adjoint situation.

Proof. Consider the following diagram

$$\begin{tikzcd} B\flat X \arrow[r, "\xi"] \arrow[d, "f_0\flat g"'] & X \arrow[d, "g"] \\ B'\flat X' \arrow[r, "\xi'_A"] & X' \end{tikzcd} \quad : \quad \begin{tikzcd} X \arrow[r, "\sigma\iota_1"] \arrow[d, "g"'] & F(X,B) \arrow[r, shift left, "\overline{[0\ 1]}"] \arrow[d, "f_1"'] & B \arrow[l, shift left, "\sigma\iota_2"] \arrow[d, "f_0"] \\ X' \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B' \arrow[l, shift left, "\beta"] \end{tikzcd} \tag{4.2}$$

where (X, ξ, B) is an object in $Act\,(\mathbf{B})$ and (X', ξ'_A, B') is $S\,(A, \alpha, \beta, B')$.

Given (f_1, f_0), since k is the kernel of α and

$$\alpha f_1\sigma\iota_1 = f_0\overline{[0\ 1]}\sigma\iota_1 = f_0[0\ 1]\iota_1 = 0$$

we obtain g as the unique morphism such that $kg = f_1\sigma\iota_1$.

To prove that the pair (g, f_0) is a morphism in $Act\,(\mathbf{B})$, that is, the left hand square in (4.2) commutes, we have to show

$$\xi'_A\,(f_0\flat g) = g\xi$$

and we do the following: first observe that $k\xi'_A\,(f_0\flat g) = f_1\sigma k''$, in fact (see diagram below, where k' and k'' are kernels)

$$\begin{tikzcd}
B\flat X \arrow[rr, "k''"] \arrow[dr, "\xi"] \arrow[dd, "f_0\flat g"'] & & X\sqcup B \arrow[rr, shift left, "{[0\ 1]}"] \arrow[dr, "\sigma"] \arrow[dd, "g\sqcup f_0"'] & & B \arrow[ll, shift left, "\sigma\iota_2"] \arrow[dr, equal] \arrow[dd, "f_0"] & \\
& X \arrow[rr, "\sigma\iota_1"] \arrow[dd, "g"] & & F(X,B) \arrow[rr, shift left, "\overline{[0\ 1]}"] \arrow[dd, "f_1"] & & B \arrow[ll, shift left, "\sigma\iota_2"] \arrow[dd, "f_0"] \\
B'\flat X' \arrow[rr, "k''"] \arrow[dr, "\xi'_A"'] & & X'\sqcup B' \arrow[rr, shift left, "{[0\ 1]}"] \arrow[dr, "{[k\ \beta]}"] & & B' \arrow[ll, shift left, "\iota_2"] \arrow[dr, equal] & \\
& X' \arrow[rr, "k"] & & A \arrow[rr, shift left, "\alpha"] & & B' \arrow[ll, shift left, "\beta"]
\end{tikzcd}$$

$$\begin{aligned} k\xi'_A\,(f_0\flat g) &= [k\ \beta]k'\,(f_0\flat g) \ \text{, definition of } \xi'_A \\ &= [k\ \beta]\,(g\sqcup f_0)\,k'' \\ &= [kg\ \beta f_0]k'' \end{aligned}$$

and

$$\begin{aligned} f_1\sigma k'' &= f_1\sigma[\iota_1\ \iota_2]k'' \\ &= [f_1\sigma\iota_1\ f_1\sigma\iota_2]k'' \\ &= [kg\ \beta f_0]k''; \end{aligned}$$

we also have $kg\xi = f_1\sigma\iota_1\xi$, by definition of g. The result follows from the fact that k is monic and

$$\sigma k'' = \sigma\iota_1\xi,$$

which follows from (4.3) by taking $f = \sigma\iota_1$ and $g = \sigma\iota_2$.

Conversely, given g and f_0 such that the left hand square in (4.2) commutes, we find $f_1 = \overline{[kg\ \beta f_0]}$, which is well defined because (see 4.3 below)

$$kg\xi = [kg\ \beta f_0]k'',$$

indeed we have

$$\begin{aligned} kg\xi &= k\xi'_A\,(f_0\flat g) \\ &= [k\ \beta]k'\,(f_0\flat g) \\ &= [k\ \beta]\,(g+f_0)\,k'' \\ &= [kg\ \beta f_0]k''. \end{aligned}$$

□

In what follows we will need the following.

To give a morphism

$$F\,(X,\xi,B) \longrightarrow B'$$

is to give a pair (f,g) with $f : X \longrightarrow B'$ and $g : B \longrightarrow B'$ such that

$$[f\ g][k\ \iota_2] = [f\ g]\,(\xi \sqcup 1)$$

or equivalently

$$[f\ g]k = f\xi. \tag{4.3}$$

See [**1**] for more details.

Theorem 13. *Let* $\mathbf{B}$ *be a pointed category with binary products and coproducts, kernels of split epis and coequalizers of reflexive graphs. If the canonical functor*

$$Act\,(\mathbf{B}) \xrightarrow{T} Pt\,(\mathbf{B})$$

is an equivalence, then:

$$RG\,(\mathbf{B}) \sim \textit{Pre-X-Mod}\,(\mathbf{B})$$

$$PC\,(\mathbf{B}) \sim \textit{2-ChainComp}\,(\mathbf{B})\,.$$

The objects in Pre-X-Mod$(\mathbf{B})$ *are pairs* (h,ξ) *with* $h : X \longrightarrow B$ *a morphism in* $\mathbf{B}$ *and* $\xi : B\flat X \longrightarrow X$ *an action* (X,ξ,B) *in* $Act\,(\mathbf{B})$ *satisfying the following condition*

$$[h\ 1]k = h\xi,$$

with $k : B\flat X \longrightarrow X + B$ *the kernel of* $[0\ 1]$;
The objects in 2-ChainComp$(\mathbf{B})$ *are sequences*

$$Z \xrightarrow{t} X \xrightarrow{h} B \quad , \quad ht = 0$$

together with actions

$$\begin{array}{rcl} \xi_X & : & B\flat X \longrightarrow X \\ \xi_Z & : & X\flat Z \longrightarrow Z \\ \xi_{F(Z,X)} & : & F(X,B)\flat F(Z,X) \longrightarrow F(Z,X) \end{array}$$

subject to the following conditions

$$\begin{array}{rcl} [h\ 1]k_X & = & h\xi_X \\ [t\ 1]k_Z & = & t\xi_Z \\ [\sigma\iota_1\overline{[t\ 1]}\ 1]k_{F(Z,X)} & = & \sigma\iota_1\overline{[t\ 1]}\xi_{F(Z,X)} \\ \overline{[0\ 1]}\xi_{F(Z,X)} & = & \xi_X\left(\overline{[h\ 1]}\flat\overline{[0\ 1]}\right) \\ \sigma\iota_2\xi_X & = & \xi_{F(Z,X)}\left(\sigma\iota_2\flat\sigma\iota_2\right). \end{array}$$

PROOF. Using the equivalence T, a reflexive graph in $\mathbf{B}$ is of the form

$$\begin{tikzcd} F(X,B) \arrow[r, shift left=2, "\overline{[0\,1]}"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "\sigma\iota_2"' description] \end{tikzcd} \qquad c\sigma\iota_2 = 1.$$

By definition of $F\,(X,B)$ we have

$$\begin{tikzcd} X \arrow[r, "\sigma\iota_1"] \arrow[dr, "h=c\sigma\iota_1"'] & F(X,B) \arrow[d, "c"] & B \arrow[l, "\sigma\iota_2"'] \arrow[dl, equal] \\ & B & \end{tikzcd}$$

and the pair $(h,1)$ induces $c=\overline{[h\ 1]}$ if and only if

$$[h\ 1]k = h\xi.$$

For a precategory, observing that a split square (2.2) is in fact a split epi in $Pt\,(\mathbf{B})$ and using the equivalence $Act\,(\mathbf{B}) \sim Pt\,(\mathbf{B})$ we have that every such split square is of the form

$$\begin{tikzcd} F(Y,F(X,B)) \arrow[r, shift left, "\overline{[0\,1]}"] \arrow[d, shift right, "F(a,\overline{[h,1]})"'] & F(X,B) \arrow[l, shift left, "\sigma\iota_2"] \arrow[d, shift right, "\overline{[h\,1]}"'] \\ F(X,B) \arrow[r, shift left, "\overline{[0\,1]}"] \arrow[u, shift right, "F(b,\sigma\iota_2)"'] & B \arrow[l, shift left, "\sigma\iota_2"] \arrow[u, shift right, "\sigma\iota_2"'] \end{tikzcd},$$

and hence giving such a split square is to give internal actions (X,ξ,B) and $(Y,\xi',F\,(X,B))$ together with morphisms a,b,h such that the following squares commute

$$\begin{tikzcd} F(X,B)\flat Y \arrow[r, "\xi'"] \arrow[d, shift right, "\overline{[h\,1]}\flat a"'] & Y \arrow[d, shift right, "a"'] & \\ B\flat X \arrow[r, "\xi"] \arrow[u, shift right, "\sigma\iota_2\flat b"'] & X \arrow[r, "h"] \arrow[u, shift right, "b"'] & B \end{tikzcd}$$

and

$$[h\ 1]k = h\xi.$$

It remains to insert the morphism

$$m : F(Y, F(X, B)) \longrightarrow F(X, B)$$

satisfying the following conditions

$$\begin{aligned}
m\sigma\iota_2 &= 1 && (4.4)\\
mF(b, \sigma\iota_2) &= 1 && (4.5)\\
\overline{[h\ 1]}m &= \overline{[h\ 1]}F\left(a, \overline{[h\ 1]}\right) && (4.6)\\
\overline{[0\ 1]}m &= \overline{[0\ 1][0\ 1]}. && (4.7)
\end{aligned}$$

From (4.4) we conclude that $m = \overline{[v\ 1]}$ for some $v : Y \longrightarrow F(X, B)$ such that

$$[v\ 1]k' = v\xi'.$$

Using (4.7) we conclude that $\overline{[0\ 1]}v = 0$ so that v factors through the kernel of $\overline{[0\ 1]}$, which is $\sigma\iota_1$ because T is an equivalence, and finally we have

$$m = \overline{[\sigma\iota_1 u\ 1]}$$

for some $u : Y \longrightarrow X$ such that

$$[\sigma\iota_1 u\ 1]k' = \sigma\iota_1 u\xi'.$$

Condition (4.5) gives $ub = 1$ while condition (4.6) gives $ha = hu$:

$$\begin{aligned}
mF(b, \sigma\iota_2) = 1 \Leftrightarrow \overline{[\sigma\iota_1 u\ 1]}F(b, \sigma\iota_2) = 1 \Leftrightarrow \overline{[\sigma\iota_1 u\ 1][\sigma\iota_1 b\ \sigma\iota_2\sigma\iota_2]} = 1 \Leftrightarrow\\
\Leftrightarrow \overline{[\sigma\iota_1 u\ 1]}[\sigma\iota_1 b\ \sigma\iota_2\sigma\iota_2] = \sigma \Leftrightarrow [\sigma\iota_1 ub\ \ \sigma\iota_2] = [\sigma\iota_1\ \sigma\iota_2] \Leftrightarrow\\
\Leftrightarrow \sigma\iota_1 ub = \sigma\iota_1 \Leftrightarrow ub = 1;
\end{aligned}$$

$$\begin{aligned}
\overline{[h\ 1]}m &= \overline{[h\ 1]}F\left(a, \overline{[h\ 1]}\right) \Leftrightarrow \overline{[h\ 1][\sigma\iota_1 u\ 1]} = \overline{[h\ 1]}\overline{[\sigma\iota_1 a\ \ \sigma\iota_2\overline{[h\ 1]}]} \Leftrightarrow\\
&\Leftrightarrow \overline{[h\ 1]}[\sigma\iota_1 u\ 1] = \overline{[h\ 1]}[\sigma\iota_1 a\ \ \sigma\iota_2\overline{[h\ 1]}] \Leftrightarrow\\
&\Leftrightarrow [hu\ \overline{[h\ 1]}] = [ha\ \overline{[h\ 1]}] \Leftrightarrow hu = ha.
\end{aligned}$$

Conclusion 1: A precategory in $\mathbf{B}$ is given by the following data

$$\begin{tikzcd}
F(X,B)\flat Y \arrow[r, "\xi'"] \arrow[d, shift right, "\overline{[h\ 1]}\flat a"'] & Y \arrow[d, shift right, "a"'] \arrow[d, bend left=60, "u"] \\
B\flat X \arrow[r, "\xi"] \arrow[u, shift right, "\sigma\iota_2\flat b"'] & X \arrow[r, "h"] \arrow[u, shift right, "b"'] & B
\end{tikzcd} \tag{4.8}$$

such that ξ,ξ' are internal actions, the obvious squares commute, and the following conditions are satisfied

$$\begin{aligned}
hu &= ha && (4.9)\\
ub &= 1 = ab\\
[\sigma\iota_1 u\ 1]k' &= \sigma\iota_1 u\xi'\\
[h\ 1]k &= h\xi.
\end{aligned}$$

We now continue to investigate it further and replace the split epi (Y, a, b, X) with an action (Z, ξ_Z, B). For convenience we will also rename $\xi_X := \xi$, $k_X := k$,

$\xi_{F(Z,X)} := \xi'$, $k_{F(Z,X)} = k'$. The diagram (4.8) becomes

$$\begin{tikzcd}
F(X,B)\flat F(Z,X) \arrow[rr, "\xi_{F(Z,X)}"] \arrow[d, shift right, "\overline{[h\,1]}\flat\overline{[0\,1]}"'] & & F(Z,X) \arrow[d, shift right, "\overline{[0\,1]}"'] \arrow[d, bend left, "\overline{[t\,1]}"] \\
B\flat X \arrow[rr, "\xi_X"] \arrow[u, shift right, "\sigma\iota_2\flat\sigma\iota_2"'] & & X \arrow[r, "h"] \arrow[u, shift right, "\sigma\iota_2"'] & B
\end{tikzcd}$$

for some $t : Z \longrightarrow X$ such that $[t\ 1]k_Z = t\xi_Z$. The commutativity of the appropriate squares in the diagram above plus the reinterpretation of conditions (4.9) gives the stated result. $\square$

Remark 2. *In order to consider a reflexive graph* $(h : X \longrightarrow B, \xi : B\flat X \longrightarrow X)$ *as a precategory of the form*

$$\begin{tikzcd}
F(X,B)\flat X \arrow[r, "\xi(\overline{[h\,1]}\flat 1)"] \arrow[d, shift right, "\overline{[h\,1]}\flat 1"'] & X \arrow[d, shift right, "1"'] \arrow[d, bend left, "1"] \\
B\flat X \arrow[r, "\xi"] \arrow[u, shift right, "\sigma\iota_2\flat 1"'] & X \arrow[r, "h"] \arrow[u, shift right, "1"'] & B
\end{tikzcd}$$

we need, in addition to $[h\ 1]k = h\xi$ *that*

$$[\sigma\iota_1\ 1]k' = \sigma\iota_1\xi\left(\overline{[h\ 1]}\flat 1\right)$$

where $k : B\flat X \longrightarrow X \sqcup B$ *and* $k' : F(X,B)\flat X \longrightarrow X \sqcup F(X,B)$ *are the kernels of* $[0\ 1]$.
In the case of Groups, this corresponds to the Peiffer identity that distinguishes a precrossed module from a crossed module.

We now give a characterization of categories $\mathbf{B}$ such that $Act(\mathbf{B}) \sim Pt(\mathbf{B})$.

Theorem 14. *Let* $\mathbf{B}$ *be a pointed category with binary products and coproducts, kernels of split epis and coequalizers of reflexive graphs. The canonical functor*

$$Act(\mathbf{B}) \xrightarrow{T} Pt(\mathbf{B})$$

is an equivalence if and only if the following two properties holds in $\mathbf{B}$*:*
(A1) for every diagram of the form

$$X \xrightarrow{\sigma\iota_1} F(X,B) \underset{\sigma\iota_2}{\overset{\overline{[0\,1]}}{\rightleftarrows}} B \tag{4.10}$$

the morphism $\sigma\iota_1$ *is the kernel of* $\overline{[0\ 1]}$*;*
(A2) the split short five lemma holds.

Proof. Similar to Theorem 9. $\square$

5. The general case

Let

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}\ ,\ \ \pi : 1 \longrightarrow GI$$

be a half-reflection.

Define a new category, denoted by $\mathbf{A}_1$ as follows:
Objects are pairs (A,u) with $A \in \mathbf{A}$ and

$$u : A \longrightarrow GIA$$

such that $I(u) = 1$.
A morphism $f : (A,u) \longrightarrow (A',u')$ is a morphism $f : A \longrightarrow A'$ in $\mathbf{A}$ such that

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & GIA \arrow[d, "GIf"] \\ A' \arrow[r, "u'"] & GIA' \end{tikzcd} .$$

Define another category, denoted $\mathbf{A}_2$, as follows:
Objects are systems

$$((E,v), a, b, (A,u))$$

where (E,v) and (A,u) are objects in $\mathbf{A}_1$,

$$a : (E,v) \longrightarrow (A,u)$$

is a morphism in $\mathbf{A}_1$, and

$$b : A \longrightarrow E$$

is a morphism in $\mathbf{A}$ such that

$$ab = 1_A.$$

Let $\mathbf{A} \xrightarrow{T} \mathrm{Pt}(\mathbf{B})$ be any subcategory of $\mathrm{Pt}(\mathbf{B})$, not necessarily full, we may consider the subcategories of reflexive graphs and internal precategories in $\mathbf{B}$, restricted to split epis in $T(\mathbf{A})$, and denote it respectively by $RG_{\mathbf{A}}(\mathbf{B})$ and $PC_{\mathbf{A}}(\mathbf{B})$.

In particular if the functor G, as above, admits a left adjoint (F,G,η,ε) and F is faithful and injective on objects, then the canonical functor

$$\mathbf{A} \xrightarrow{T} Pt(\mathbf{B})$$

determines a subcategory of split epis and so we have:
Reflexive graphs internal to $\mathbf{B}$ and restricted to the split epis in $T(\mathbf{A})$, denoted $RG_{\mathbf{A}}(\mathbf{B})$ as follows:

$$\begin{tikzcd} FA \arrow[r, shift left=2, "\varepsilon_{IA}F(\pi_A)"] \arrow[r, shift right=2, "c"'] & IA \arrow[l, "I(\eta_A)" description] \end{tikzcd} \quad , \quad cI(\eta_A) = 1_{IA};$$

for some $A \in \mathbf{A}$.
Internal precategories in $\mathbf{B}$ relative to split epis in $T(\mathbf{A})$, denoted by $PC_{\mathbf{A}}(\mathbf{B})$, as follows (where we use π'_A as an abbreviation to $\varepsilon_{IA}F(\pi_A)$ and similarly to π'_E)

$$\begin{tikzcd} F(E) \arrow[r, shift left=4, "\pi'_E"] \arrow[r, "m" description] \arrow[r, shift right=4, "F(a)"'] & IE = FA \arrow[l, shift left=2, "I(\eta_E)" description] \arrow[l, shift right=2, "F(b)" description] \arrow[r, shift left=2, "\pi'_A"] \arrow[r, shift right=2, "c"'] & IA \arrow[l, "I(\eta_A)" description] \end{tikzcd} \tag{5.1}$$

for some

$$\begin{tikzcd} E \arrow[r, shift left, "a"] & A \arrow[l, shift left, "b"] \end{tikzcd} \quad , \quad ab = 1_A$$

in $\mathbf{A}$, and satisfying the following conditions

$$cI(\eta_A) = 1_{IA} \tag{5.2}$$

$$I(a) = c \tag{5.3}$$

$$I(b) = I(\eta_A) \tag{5.4}$$

$$mI(\eta_E) = 1_{IE} \tag{5.5}$$

$$mF(b) = 1_{IE} \tag{5.6}$$

$$cm = cF(a) \tag{5.7}$$

$$\pi'_A m = \pi'_A \pi'_E. \tag{5.8}$$

We observe that c is determined by a, and (5.2) follows from (5.3), (5.4) and the fact that $ab = 1_A$. We will be also interested in the notion of multiplicative graph, which is obtained by removing (5.7) and (5.8) and in some cases we may be also interested in removing (5.6) so that the definition may be transported from $\mathbf{B}$ to $\mathbf{A}$ and it does not depend on whether or not G admits a left adjoint.

Theorem 15. *For a half-reflection*

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \quad , \quad \pi : 1 \longrightarrow GI,$$

if the functor G admits a left adjoint

$$(F, G, \eta, \varepsilon),$$

and F is faithful and injective on objects, then

$$\mathbf{A}_1 \cong RG_{\mathbf{A}}(\mathbf{B}) \tag{5.9}$$

$$\mathbf{A}_2^* \cong PC_{\mathbf{A}}(\mathbf{B}) \tag{5.10}$$

where $\mathbf{A}_2^*$ is the subcategory of $\mathbf{A}_2$ given by the objects

$$((E, v), a, b, (A, u))$$

such that

$$\begin{aligned} IE &= FA \\ I(a) &= \varepsilon_{IA} F(u) \\ vb &= \eta_A \\ G(\varepsilon_{IA} F(\pi_A)) \pi_E &= G(\varepsilon_{IA} F(\pi_A)) v. \end{aligned}$$

Proof. The isomorphism (5.9) is established by the adjuntion $(F, G, \eta, \varepsilon)$. Given

$$A \xrightarrow{u} GIA \quad , \quad I(u) = 1_{IA}$$

we obtain

$$\begin{tikzcd} FA \arrow[r, shift left=2, "\pi'_A"] \arrow[r, shift right=2, "u'"'] & IA \arrow[l, dashed, "I(\eta_A)" description] \end{tikzcd} \tag{5.11}$$

where $\pi'_A = \varepsilon_{IA} F(\pi_A)$, $u' = \varepsilon_{IA} F(u)$ and

$$u' I(\eta_A) = 1_{IA} \Leftrightarrow I(u) = 1.$$

Conversely, given (5.11), we obtain A, since F is injective on objects, and

$$u = G(u') \eta_A.$$

The isomorphism (5.10) is obtained as follows:
Given (5.1), since F in injective on objects and faithful, we obtain

$$\begin{tikzcd} E \arrow[r, shift left, "a"] & A \arrow[l, shift left, "b"] \end{tikzcd} \quad , \quad ab = 1_A \quad , \quad IE = FA.$$

Now define on the one hand

$$u = G(c)\,\eta_A \quad , \quad v = G(m)\,\eta_E;$$

while on the other hand

$$c = \varepsilon_{IA} F(u) \quad , \quad m = \varepsilon_{IE} F(v),$$

and we have the following translation of equations

Eq. n.^{o}	in $\mathbf{B}$	in $\mathbf{A}$
5.2	$cI(\eta_A) = 1_{IA}$	$I(u) = 1_{IA}$
5.5	$mI(\eta_E) = 1_{IE}$	$I(v) = 1_{IE}$
5.7	$cm = cF(a)$	$ua = GI(a)\,v$
5.3	$I(a) = c$	$I(a) = \varepsilon_{IA} F(u)$
5.4 5.6	$I(b) = I(\eta_A)$ $mF(b) = 1_{IE}$	$vb = \eta_A$
5.8	$\pi'_A m = \pi'_A \pi'_E$	$G(\pi'_A)\,\pi_E = G(\pi'_A)\,v$

Note that $ua = GI(a)\,v$ follows from the fact that a is a morphism in $\mathbf{A}_1$, on the contrary of b which is simply a morphism in $\mathbf{A}$. □

In some cases we also have a functor

$$J : \mathbf{A} \longrightarrow \mathbf{B}$$

satisfying the following three conditions:

(1) $JG = 1_{\mathbf{B}}$
(2) the pair $(J(\eta_A), I(\eta_A))$ is jointly epic for every $A \in \mathbf{A}$, that is, given a pair of morphisms (f, g) as displayed below

$$\begin{tikzcd} JA \arrow[r, "J(\eta_A)"] \arrow[dr, "f"'] & FA \arrow[d, dashed, "{[f\ g]}"] & IA \arrow[l, "I(\eta_A)"'] \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

there is at most one morphism $\alpha : FA \longrightarrow B$, with the property that $\alpha J(\eta_A) = f$ and $\alpha I(\eta_A) = g$, denoted by $\alpha = [f, g]$ when it exists. Also the pair (f, g) is said to be admissible (or cooperative in the sense of Bourn and Gran [**6**]) w.r.t. A, if $[f, g]$ exists.
(3) for every $A, E \in \mathbf{A}$, with $IE = FA$, a morphism, $u : J(E) \longrightarrow FA$, such that $(u, 1_{IE})$ is cooperative w.r.t. E and satisfying $\pi_A[u\ 1] = \pi_A \pi_E$, always factors trough $J(A)$, i.e., given u as in the diagram below

$$\begin{tikzcd} JE \arrow[r, "J(\eta_E)"] \arrow[d, dashed, "\bar{u}"'] \arrow[dr, "u"] & FE \arrow[r, shift left, "\pi'_E"] \arrow[d, dotted, "{[u\,1]}"] & IE \arrow[l, shift left, dashed, "I(\eta_E)"] \arrow[dl, equal] \\ JA \arrow[r, "J(\eta_A)"'] & FA \arrow[r, "\pi'_A"'] & IA \end{tikzcd}$$

such that $[u\ 1]$ exists and $\pi_A[u\ 1] = \pi_A\pi_E$ then $u = J(\eta_A)\,u'$ for a unique $u' : JE \longrightarrow JA$.

THEOREM 16. *Let* $\mathbf{B}$ *be any category, with* (I, G, π)

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift left, "I"] & \mathbf{B} \arrow[l, shift left, "G"] \end{tikzcd} \quad , \quad \pi : 1 \longrightarrow GI,$$

a half-reflection such that the functor G *admits a left adjoint*

$$(F, G, \eta, \varepsilon)\,.$$

If we can find a functor

$$J : \mathbf{A} \longrightarrow \mathbf{B}$$

as above, then

- *the category* $RG_{\mathbf{A}}(\mathbf{B})$ *of reflexive graphs in* $\mathbf{B}$ *relative to split epis from* $\mathbf{A}$*, is given by:*
  Objects are pairs (A, h)*, with* $A \in \mathbf{A}$*, and* $h : JA \longrightarrow IA$ *a morphism such that* $(h, 1_{IA})$ *is cooperative w.r.t.* A*;*
  A morphism $f : (A, h) \longrightarrow (A', h')$ *is a morphism* $f : A \longrightarrow A'$ *in* $\mathbf{A}$ *such that* $h'F(f) = I(f)\,h$*.*
- *the category of internal precategories in* $\mathbf{B}$ *relative to split epis from* $\mathbf{A}$*,* $PC_{\mathbf{A}}(\mathbf{B})$*, is given by:*
  Objects:

  $$(A, E, a, b, t, h)$$

  where A, E*, are objects in* $\mathbf{A}$ *, with* $IE = FA$*,* a, b, t, h*, are morphisms in* $\mathbf{B}$*,*

  $$\begin{tikzcd} JE \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "t"'] & JA \arrow[l, "b" description] \arrow[r, "h"] & IA \end{tikzcd}$$

  such that

  $$ab = 1 = tb \quad , \quad ha = ht$$

  and

$$\begin{array}{lll} (h, 1_{IA}) & \textit{and } \left(J(\eta_E)\,b, I(\eta_E)\,I(\eta_A)\right) & \textit{are cooperative w.r.t. } A \\ \left(J(\eta_A)\,a, I(\eta_A)\,[h\ 1]\right) & \textit{and } \left(J(\eta_A)\,t, 1_{I(E)}\right) & \textit{are cooperative w.r.t. } E \end{array}$$

  Morphisms are triples (f_3, f_2, f_1) *of morphisms*

  $$\begin{tikzcd} JE \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "t"'] \arrow[d, "f_3"] & JA \arrow[l, "b" description] \arrow[r, "h"] \arrow[d, "f_2"] & IA \arrow[d, "f_1"] \\ JE' \arrow[r, shift left=2, "a'"] \arrow[r, shift right=2, "t'"'] & JA' \arrow[l, "b'" description] \arrow[r, "h'"] & IA' \end{tikzcd}$$

  such that the obvious squares in the above diagram commute and furthermore the pair $(J(\eta_{A'})\,f_2, I(\eta_{A'})\,f_1)$ *is admissible w.r.t.* A *while the pair* $(J(\eta_{E'})\,f_3, I(\eta_{E'})\,[J(\eta_{A'})\,f_2\ \ I(\eta_{A'})\,f_1])$ *is admissible w.r.t.* E*.*

Proof. Calculations are similar to the previous sections and the resulting diagram (5.1) is given by

$$\begin{aligned} c &= [h\ 1] \\ m &= [J(\eta_A)\, t\ \ 1_{I(E)}] \\ F(b) &= [J(\eta_E)\, b\ \ I(\eta_E)\, I(\eta_A)] \\ F(a) &= [J(\eta_A)\, a\ \ I(\eta_A)\, [h\ 1]]. \end{aligned}$$

The same argument applies to obtain the morphisms. □

5.1. The example of unitary magmas with right cancellation. An example of a general situation in the conditions of the above theorem is the following one.

Let $\mathbf{B}$ be a pointed category with kernels of split epis, with binary products and coproducts and such that the pair $(\langle 1,0\rangle, \langle 1,1\rangle)$, as displayed

$$B \xrightarrow{<1,0>} B \times B \underset{<1,1>}{\overset{\pi_2'}{\rightleftarrows}} B$$

is jointly epic for every $B \in \mathbf{B}$, and then consider: $\mathbf{A}$, the full subcategory of $\mathrm{Pt}(\mathbf{B})$ given by the split epis with the property that

$$X \xrightarrow{\ker \alpha} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the pair $(\ker \alpha, \beta)$ is jointly epic (identifying (A, α, β, B) with (A', α', β', B) whenever $A \cong A'$, in order to obtain F injective on objects).

Then we have functors

$$\begin{aligned} I, F, J &: \mathbf{A} \longrightarrow \mathbf{B} \\ G &: \mathbf{B} \longrightarrow \mathbf{A} \end{aligned}$$

with

$$\begin{aligned} I(A, \alpha, \beta, B) &= B \\ F(A, \alpha, \beta, B) &= A \\ J(A, \alpha, \beta, B) &= X \quad \text{, the object kernel of } \alpha \\ G(B) &= (B \times B, \pi_2, \langle 1,1\rangle, B) \end{aligned}$$

and with $\pi : 1_{\mathbf{A}} \longrightarrow GI$ given by $\pi = [0\ 1]$.

An example of such a category is the category of unitary magmas with right cancellation. Also every strongly unital category satisfies the above requirements (see [**8**], and references there).

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